Only rational homology spheres admit $\Omega(f)$ to be union of DE attractors

Fan Ding, Jianzhong Pan, Shicheng Wang and Jiangang Yao

Abstract

If there exists a diffeomorphism f on a closed, orientable n-manifold M such that the non-wandering set $\Omega(f)$ consists of finitely many orientable ($\pm$) attractors derived from expanding maps, then M is a rational homology sphere; moreover all those attractors are of topological dimension $n-2$.

Expanding maps are expanding on (co)homologies.

Contents

1 Introduction

1.1 Results, motivations and outline of the proof

Hyperbolic attractors derived from expanding maps, which are also called as (generalized) Smale solenoids or solenoid attractors, are introduced into dynamics by Smale in his celebrated paper [Sm]. For a diffeomorphism f with Axiom A, which has fundamental importance in the study of various stabilities in dynamics, Smale proved the Spectral Decomposition Theorem, which states that $\Omega(f)$, the non-wandering set of f, can be decomposed into the so-called *basic sets*. He posed several types of basic sets: *Group 0* which are zero dimensional ones such as isolated points or the Smale horseshoes; *Group A* and *Group DA*, both of which are derived from Anosov maps; and *Group DE* which are attractors derived from expanding maps.

In the same paper [Sm], Smale posed the following conjecture for Anosov map.

Conjecture: *If $f : M \to M$ is an Anosov diffeomorphism, then the non-wandering set $\Omega(f) = M$.*

For Anosov flows on 3-manifolds, the corresponding problem has a negative answer given by Franks and Williams [FW]. Inspired by the construction of [FW], experts in dynamics recognized if one can construct a dynamics $f : M \to M$ on $2k$-manifold M such that $\Omega(f)$ consists of one attractor and one repeller which are derived from expanding maps of type (k, k) (cf. Definitions 1.6 and 1.7, a repeller of f is an attractor of f^{-1}) with additional transversality condition on stable and unstable manifolds, the conjecture will also have a negative answer. This provides a direct motivation to prove the following Theorem 1.1, which implies no dynamics f can make $\Omega(f)$ to be one attractor and one repeller derived from expanding maps of type (k, k), at least in the orientable category (Corollary 1.2 (c)).

A basic and elementary fact for a diffeomorphism f on a closed manifold is that, if the non-wondering set $\Omega(f)$ consists of finitely many attractors and repellers derived from expanding maps, then $\Omega(f)$ must consist of exactly one attractor and one repeller (see [JNW] Lemma 1 for a proof). For brevity we often call attractors and repellers derived from expanding maps as $(\pm)$ attractors derived from expanding maps.

Theorem 1.1. *If there exists a diffeomorphism $f : M \to M$ on a closed, oriented n-manifold M such that $\Omega(f)$ consists of finitely many oriented $(\pm)$ attractors derived from expanding maps, then M is a rational homology sphere. Moreover all those attractors are of type $(n-2, 2)$.*

Corollary 1.2. *Let M be a closed oriented n-manifold. If there is a diffeomorphism $f : M \to M$ such that $\Omega(f)$ consists of finitely many oriented $(\pm)$ attractors derived from expanding maps, then*

(a) the dimension n of the manifold cannot be 4;

(b) if the dimension n is greater than 4, those $(\pm)$ attractors cannot be both derived from expanding maps on tori;

(c) those attractors cannot be both of type (k, k).

There is a diffeomorphism f on a rational homology 3-sphere such that $\Omega(f)$ consists of finitely many $(\pm)$ attractors of type $(1, 2)$ derived from expanding maps. Indeed recently [JNW] showed that there exists a diffeomorphism $f : M \to M$ on a closed orientable 3-manifold M with $\Omega(f)$ a union of finitely many $(\pm)$ attractors derived from expanding maps if and only if M is a lens space $L(p, q)$.

We would like to thank Professors L. Wen and C. Bonatti for passing the information around Smale's conjecture for Anosov maps once they heard the work [JNW] and asking us if there is a diffeomorphism $f : M \to M$ on a closed 4-manifold M such that $\Omega(f)$ consists of two $(\pm)$ attractors derived from expanding maps of type $(2, 2)$.

A more primary motivation for working on Theorem 1.1 and [JNW] is that the information of hyperbolic non-wandering set $\Omega(f)$ should provide topological information of the manifold M. Both Theorem 1.1 and results in [JNW] can be considered as an analog or a generalization of the classical result in mathematics that if there exists a diffeomorphism $f : M \to M$ on a closed, orientable n-manifold M with $\Omega(f)$ a union of finitely many isolated attractors, then M must be the n-sphere.

Stimulated by [JNW] and Corollary 1.2 (a), it is natural to ask

Question 1.3. *For which positive integer n, there is a dynamics $f : M \to M$ on a closed n-manifold M with $\Omega(f)$ a union of finitely many $(\pm)$ attractors derived from expanding maps?*

To prove Theorem 1.1, besides to use ideas and methods from algebraic and geometric topology, as well as dynamics, we also need the following result, which is of independent interests.

Theorem 1.4. *Let $f : X \to X$ be an expanding map on the closed oriented manifold X, then the induced homomorphisms $f_* : H_l(X, \mathbb{R}) \to H_l(X, \mathbb{R})$ and $f^* : H^l(X, \mathbb{R}) \to H^l(X, \mathbb{R})$ are both expanding for any positive integer l.*

Theorem 1.4 is proved in §2 based on two theorems by Gromov and by Nomizu respectively: any expanding map is conjugate to an infra-nil-automorphism (Theorem 2.7); for each nilmanifold N, its De Rham cohomology is isomorphic to the cohomology of the Chevalley-Eilenberg complex associated with the Lie algebra of the simply connected nilpotent Lie group covering N (Theorem 2.8).

Theorem 1.1 is proved in §3 with the outline as follows. It is known that if $\Omega(f)$ consists of finitely many ($\pm$) attractors derived from expanding maps, then $\Omega(f)$ must consist of two such attractors S_1 and S_2 with

$$S_1 = \bigcap_{h=1}^{\infty} f^h(N_1),\ S_2 = \bigcap_{h=1}^{\infty} f^{-h}(N_2),$$

where $N_1 \cong X_1^{p_1} \tilde{\times} D^{q_1}$ and $N_2 \cong X_2^{p_2} \tilde{\times} D^{q_2}$ are the defining disk bundles of S_1 and S_2 respectively. We may choose N_1 and N_2 so that

$$M = N_1 \cup N_2,\ P := N_1 \cap N_2 \text{ with } \partial P = \partial N_1 \cup \partial N_2.$$

Consider the Mayer-Vietoris long exact sequence for the pair (N_1, N_2) :

$$H_l(P) \xrightarrow{\varphi_l=(r_1,r_2)} H_l(N_1) \oplus H_l(N_2) \xrightarrow{\psi_l=s_1-s_2} H_l(M).$$

Since $f|N_1$ descends to an expanding map on X_1, $(f|N_1)_*$ is expanding on $H_l(N_1)$ by Theorem 1.4. While f_* must be an automorphism of $H_l(M)$, thus s_1 must be zero (Lemma 3.2). Similarly s_2 is zero by considering f^{-1}, and therefore ψ_l must be zero. Consequently φ_l is surjective (Lemma 3.11).

To find the obstruction for the surjectivity of φ_l, we first show that the two maps $H_l(\partial N_1) \to H_l(P) \leftarrow H_l(\partial N_2)$ induced by the inclusions are isomorphisms (Lemma 3.10, which is based on Lemma 3.1 and Lemma 3.9).

The disc bundle defining attractor S_j must have zero Euler class, and with the help of the Gysin sequence, we see that $H_l(\partial N_j) \cong H_l(X_j) \oplus H_{l-q_j+1}(X_j)$, and there is a subspace U of $H_l(\partial N_j)$ with $\dim U = \dim H_{l-q_j+1}(X_j)$ which is generated by S^{q_j-1}-bundles over cycles in X_j, $j = 1, 2$ (Lemma 3.7).

Note that both N_1 and N_2 are $K(\pi, 1)$ spaces by Theorem 2.6. If $q_1 > 2$ or $q_2 > 2$, the image of above mentioned subspace U under φ_l must be zero (Lemma 3.8). Then simple dimension analysis indicates that φ_l cannot be surjective. Therefore $q_1 = q_2 = 2$ (cf. Remark 1.8) and the attractors must be of type $(n-2, 2)$. Finally one concludes that M is a rational homology sphere by detailed homological argument.

1.2 Definitions related to DE attractors

Definition 1.5. Let M be a closed manifold, and $f : M \to M$ be a map. An *invariant set* of f is a subset $\Lambda \subset M$ such that $f(\Lambda) = \Lambda$. A point $x \in M$ is *non-wandering* if for any neighborhood U of x, $f^n(U) \cap U \neq \emptyset$ for infinitely many integers n. Then $\Omega(f)$, the *non-wandering set* of f, defined as the set of all non-wandering points, is an f-invariant closed set. A set $\Lambda \subset M$ is an *attractor* if there exists a closed neighborhood U of Λ such that $f(U) \subset \mathrm{Int}U$, $\Lambda = \bigcap_{n=1}^{\infty} f^n(U)$, and $\Lambda = \Omega(f|U)$.

Now assume M is a Riemannian manifold. A closed invariant set Λ of f is *hyperbolic* if there is a continuous f-invariant splitting of the tangent bundle $TM|_\Lambda$ into *stable* and *unstable bundles* $E^s_\Lambda \oplus E^u_\Lambda$ with

$$\|Df^m(v)\| \leq C\lambda^{-m}\|v\|, \quad \forall\, v \in E^s_\Lambda,\ m > 0,$$
$$\|Df^{-m}(v)\| \leq C\lambda^{-m}\|v\|, \quad \forall\, v \in E^u_\Lambda,\ m > 0,$$

for some fixed constants $C > 0$ and $\lambda > 1$.

A diffeomorphism f is called *Anosov* if f is hyperbolic in the whole manifold M. A map f is called *expanding* if f is Anosov and $\dim E^u_\Lambda = \dim M$, in other words, there are constants $C > 0$ and $\lambda > 1$ such that

$$\|Df^m(v)\| \geq C\lambda^m\|v\|, \quad \forall v \in TM,\ m > 0.$$

Although a metric on M is necessary to define hyperbolic, Anosov and expanding maps, whether a map falls into these categories does not depend on the choice of the metrics because all the norms on Euclidean spaces are equivalent. It is also shown in [Ma] that for any expanding map f, M has a metric for which $C = 1$.

Definition 1.6. Let X be the p-dimensional compact manifold, D^q be the q-dimensional unit disk, and $N = X \tilde{\times} D^q$, a disk bundle over X. Let $\pi : X \tilde{\times} D^q \to X$ be the projection.

Suppose an embedding $e : X \tilde{\times} D^q \to X \tilde{\times} D^q$ satisfies the following two conditions:

(1) For some expanding map $\varphi : X \to X$, the diagram

$$\begin{tikzcd} X \tilde{\times} D^q \arrow[r, "e"] \arrow[d, "\pi"'] & X \tilde{\times} D^q \arrow[d, "\pi"] \\ X \arrow[r, "\varphi"'] & X \end{tikzcd}$$

commutes, i.e. $\varphi \circ \pi = \pi \circ e$, or e preserves the disk fibers and descends to the expanding map φ by π,

(2) e shrinks the D^q factor evenly by some constant $0 < \lambda < 1$, i.e. the image of each fiber D^q under e is q-dimensional disk with radius λ,

then we call e a *hyperbolic bundle embedding* and $S = \bigcap_{n \geq 0}^{\infty} e^n(N)$ an *attractor of type* (p, q) derived from expanding map φ.

Definition 1.7. Let M be a $(p+q)$-manifold and $f : M \to M$ be a diffeomorphism. If there is an embedding $N \cong X^p \tilde{\times} D^q \subset M$ such that $f|N$ (resp. $f^{-1}|N$) conjugates $e : N \to N$, where e and N are as given in Definition 1.6, we call $S = \bigcap_{h=1}^{\infty} f^h(N)$ (resp. $S = \bigcap_{h=1}^{\infty} f^{-h}(N)$) an $(+)$ *attractor (resp.* $(-)$ *attractor) of type* (p, q) *derived from expanding maps* , and we also say N is a *defining disk bundle* of S.

For simplicity, call both $(+)$ attractor and $(-)$ attractor derived from expanding maps as *DE attractors*. Call a DE attractor is *oriented* if the base manifold X is oriented. Call a DE attractor is *toric* if X is a torus.

Remark 1.8. DE attractors of type (p, q) is homeomorphic to so called p*-dimensional solenoids* for any q. DE attractors of type (1,2), nested intersections of solid tori $S^1 \times D^2$, is the traditional solenoids introduced by Vietoris in 1927 [Vi]. The DE attractors given in Smale's paper [Sm] are of type $(p, p + 1)$, where the condition $q = p + 1$ guarantees any expanding map on base manifold can be lifted to an embedding of the disk bundle in the construction. On the other hand, DE attractors of type $(p, 1)$ do not exist for any p: there is no embedding $e : X \times D^1 \to X \times D^1$ lifted from an expanding $\varphi : X \to X$; indeed no p-dimensional solenoids embed into $(p + 1)$-dimensional manifolds [JWZ].

Recall that two maps $f : X \to X$ and $g : Y \to Y$ are called *topological conjugate* if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$.

Except noted otherwise, all homology and cohomology groups appeared in this paper are over the field of real numbers $\mathbb{R}$, i.e. $H_*(X)$ means $H_*(X; \mathbb{R})$, $H^*(X)$ means $H^*(X; \mathbb{R})$. The i-th Beti number, i.e. $\dim H_i(X; \mathbb{R})$ is denoted by $\beta_i(X)$.

2 Expanding maps

2.1 Expanding linear maps

Definition 2.1. Let V be a finite dimensional vector space over $\mathbb{R}$, $f : V \to V$ is a linear map, if all the eigenvalues of f are greater than 1 in absolute value, then f

is called *expanding*. For a real square matrix A, if all of its eigenvalues are greater than 1 in absolute value, then A is called *expanding*.

Lemma 2.2. *Let V and W be two finite dimensional real vector spaces, f, g and h are linear maps that make the following diagram commute.*

$$\begin{array}{ccc} V & \xrightarrow{f} & V \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h} \\ W & \xrightarrow[g]{} & W \end{array}$$

If f is expanding and h is surjective, then g is expanding.

Proof. Suppose $\dim V = n$, $\dim W = m$. Let $w_1, \cdots, w_m$ be a basis for W, by the surjectivity of h, choose $v_1, \cdots, v_m \in V$ such that $h(v_i) = w_i$, $1 \le i \le m$. Clearly $v_1, \cdots, v_m$ are linearly independent, we may then find more vectors $v_{m+1}, \cdots, v_n$ in $\ker h$ so that $v_1, \cdots, v_n$ is a basis for V. Suppose $f(v_i) = \sum_{j=1}^n c_{ij} v_j$. Then for $m + 1 \le i \le n$, we have

$$0 = g(0) = g(h(v_i)) = h(f(v_i)) = h(\sum_{j=1}^{n} c_{ij} v_j) = \sum_{j=1}^{m} c_{ij} w_j.$$

Thus $c_{ij} = 0$ for $m + 1 \le i \le n$ and $1 \le j \le m$. Under the above two bases, the matrix representation of f is $A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}$, where A_{11} and A_{22} are $m \times m$ and $(n - m) \times (n - m)$ matrices respectively. This matrix A is expanding, so all of its eigenvalues are greater than 1 in absolute value. Note that all the eigenvalues of A_{11} are eigenvalues of A. Hence A_{11} is expanding. It follows from $h(v_i) = w_i$ that linear map g corresponds to matrix A_{11}. Therefore g is an expanding map. □

Lemma 2.3. *Let $f : V \to V$ be a linear map on the finite dimensional real vector space V. Denote the dual map of f by $f^* : V^* \to V^*$, where $V^* = \mathrm{Hom}(V, \mathbb{R})$.*

If f is expanding, so is f^* and vice versa.

Proof. Fix a basis for V and a dual basis for V^*, then f and f^* are represented by two square matrices A and A^T respectively. The claim then follows from the fact that A and A^T have the same characteristic polynomials, and thus the same eigenvalues. □

Lemma 2.4. *Let $f : V \to V$ be an expanding linear map on the real vector space V, for any positive integer i,*

(a) the induced map $\otimes^i f : \otimes^i V \to \otimes^i V$ is expanding,

(b) the induced map $\wedge^i f : \wedge^i V \to \wedge^i V$ is expanding.

Proof. (a) There is a standard fact in multilinear algebra.

Let $f : V \to V$ and $g : W \to W$ be linear maps with $\dim V = n$ and $\dim W = m$. Suppose $\lambda_1, \cdots, \lambda_n$ are the eigenvalues of f and $\mu_1, \cdots, \mu_m$ are the eigenvalues of g, then the eigenvalues of $f \otimes g : V \otimes W \to V \otimes W$ are $\lambda_i \mu_j$, where $i = 1, \cdots, n$, and $j = 1, \cdots, m$.

Thus each eigenvalue of $\otimes^i f$ is a product of i complex numbers with absolute value greater than 1, which is still greater than 1 in absolute value. Therefore, by definition, $\otimes^i$ is expanding.

(b) Note that there is a surjective map from $\otimes^i V$ to $\wedge^i V$, then it follows from Lemma 2.2 that $\wedge^i f$ is also expanding. $\square$

Lemma 2.5. *Let (C_*, d) be a cochain complex, where each C_i is a finite dimensional real vector space, and $f : (C_*, d) \to (C_*, d)$ be a chain map. Suppose $f_i : C_i \to C_i$ is expanding for all positive integer i, then f induces expanding maps on the cohomology groups $H^i(C)$ for $i > 0$.*

Proof. We label the boundary maps in the cochain complex as $d_i : C_i \to C_{i+1}$, then

$$H^i(C) = \frac{\text{Ker } d_i}{\text{Im } d_{i-1}}.$$

As f is a chain map, Ker d_i is a f_i-invariant subspace of C_i.

Fix a positive integer i, choose a basis for Ker d_i and then extend it to a basis for C_i. Clearly, under this basis, the matrix representation of f_i takes the form $A = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$. Since f is expanding, all the eigenvalues of A are greater than 1 in absolute value, so are eigenvalues of A_{11}. Therefore, the restriction of f_i on Ker d_i is expanding. Now we have a commutative diagram

$$\begin{array}{ccc} \text{Ker } d_i & \xrightarrow{f_i} & \text{Ker } d_i \\ \downarrow & & \downarrow \\ H^i(C) & \xrightarrow{f_i^*} & H^i(C) \end{array}$$

Clearly, the two vertical maps are surjective, then according to Lemma 2.2, f_i^* on $H^i(C)$ is expanding as well. $\square$

2.2 Gromov's theorem and Nomizu's theorem

The most basic manifolds that admit expanding maps are n-tori. Expanding maps are systematically discussed in late 1960's. We quote some major properties as follows.

Theorem 2.6. *Let M be a closed n-manifold and $f : M \to M$ be an expanding map. Then*

(a) The universal cover of M is $\mathbb{R}^n$ and f is a covering map (Shub [Sh1]).

(b) Any flat manifold admits expanding maps (Epstein-Shub [ES]).

(c) If $g : M \to M$ is an expanding map which is homotopic to f, then f and g are topologically conjugate (Shub [Sh2]).

(d) $\pi_1(M)$ has polynomial growth (Franks [Sh2]).

In [Gr], Gromov proved that if a finitely generated group G has polynomial growth, then G contains a nilpotent subgroup of finite index. Combining this result with Theorem 2.6 (c) and (d), he obtained

Theorem 2.7. *An expanding map on a closed manifold is topologically conjugate to an infra-nil-endomorphisms.*

Recall that a *nilpotent* Lie group G is a connected Lie group whose Lie algebra $\mathfrak{g}$ is a *nilpotent* Lie algebra. That is, its Lie algebra lower central series

$$\mathfrak{g}_1 = [\mathfrak{g}, \mathfrak{g}], \ \mathfrak{g}_2 = [\mathfrak{g}, \mathfrak{g}_1], \cdots, \mathfrak{g}_{k+1} = [\mathfrak{g}, \mathfrak{g}_k], \ \cdots$$

eventually vanishes.

Suppose G be a simply connected nilpotent Lie group. Denote by $\mathrm{Aff}(G)$ the group of transformations of G generated by all the left translations and all the automorphisms on G. In other words, $\mathrm{Aff}(G) = G \ltimes \mathrm{Aut}(G)$.

For a discrete subgroup Γ of G, call G/Γ a *nilmanifold* when it is compact. For a group $\Gamma \subset \mathrm{Aff}(G)$ which acts freely and discretely on G, call $N = G/\Gamma$ *infra-nil-manifold* when it is compact. Every automorphism of G which respects the group action Γ induces a map on G/Γ. Such maps are called *infra-nil-endomorphism.*

Theorem 2.8. *For each nilmanifold $N = G/\Gamma$, the De Rham cohomology of N is isomorphic to the cohomology of the Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, \delta)$ associated with the Lie algebra $\mathfrak{g}$ of G, that is*

$$H^*(N; \mathbb{R}) \cong H^*(\wedge\mathfrak{g}^*, \delta).$$

In [CE], Chevalley-Eilenberg complex $(\wedge\mathfrak{g}^*, \delta)$ was constructed for Lie algebra $\mathfrak{g}$ as follows. Let $\{X_1, \cdots, X_s\}$ be the basis for $\mathfrak{g}$, and $\{x_1, \cdots, x_s\}$ be the dual basis for $\mathfrak{g}^*$, the dual of $\mathfrak{g}$. On the exterior algebra $\wedge\mathfrak{g}^*$, construct the differential δ by defining it on degree 1 elements as

$$\delta x_k(X_i, X_j) = -x_k([X_i, X_j]),$$

and extending to $\wedge\mathfrak{g}^*$ as a graded bilinear derivation. Suppose $[X_i, X_j] = \sum_l c_{ij}^l X_l$, where c_{ij}^l are the structure constants of $\mathfrak{g}$, then $\delta x_k(X_i, X_j) = -c_{ij}^k$, and thus the differential δ on generators has the form

$$\delta x_k = -\sum_{i<j} c_{ij}^k x_i \wedge x_j.$$

Now it is clear that the Jacobi identity in the Lie algebra is equivalent to the condition $\delta^2 = 0$. Hence $(\wedge\mathfrak{g}^*, \delta)$ is a cochain complex.

For Lie group G, we consider the cochain complex consisting of all the G-right invariant differential forms on G over $\mathbb{R}$ with ordinary differential d, denote it by $(C(G), d)$. From the definition of Lie algebra associated with a Lie group, we know

$$(\wedge\mathfrak{g}^*, \delta) \cong (C(G), d).$$

For $N = G/\Gamma$, Let $C(N)$ be the cochain complex consisting of all the Γ-right invariant differential forms on G over $\mathbb{R}$, denote it by $(C(N), d)$. It is clear that $H^*(C(N))$ is the De Rham cohomology of the nilmanifold N, that is

$$H^*(N; \mathbb{R}) \cong H^*(C(N))$$

Under the condition that G is nilpotent, Nomizu [No] showed that the natural inclusion $C(G) \to C(N)$ induces an isomorphism on the cohomology level

$$H^*(C(N)) \cong H^*(C(G))$$

Theorem 2.8 then follows from the above three isomorphisms.

2.3 Expanding maps are expanding on (co)homology groups

Now we are ready to prove

Theorem 1.4 *Let $f : X \to X$ be an expanding map on the closed oriented manifold X, then the induced homomorphisms $f_* : H_l(X, \mathbb{R}) \to H_l(X, \mathbb{R})$ and $f^* : H^l(X, \mathbb{R}) \to H^l(X, \mathbb{R})$ are both expanding for any positive integer l.*

Before the proof, we need a lemma about the induced homomorphisms on homology groups of covering and expanding maps.

Lemma 2.9. *(a) If $f : X \to Y$ is a covering map between the oriented closed manifolds, then the induced map $f_* : H_*(X) \to H_*(Y)$ is an epimorphism.*

(b) If f is an expanding map on the oriented closed manifold X, then the induced map $f_* : H_*(X) \to H_*(X)$ is an isomorphism.

Proof. (a) $f : X \to Y$ is a covering between the oriented closed manifolds implies that f is a map of non-zero degree. Then it is well-known that f_* is surjective (cf. Lemma 7 in [WZ]).

(b) $f : X \to X$ is expanding implies that f is a covering. X is closed and orientable implies that f is a map of non-zero degree. Hence f_* is surjective on $H_*(X)$ by (a). A self-surjection on any finite dimensional real vector space must be an isomorphism. $\square$

Proof of Theorem 1.4. As topologically conjugate maps induce conjugate homomorphisms on (co)homology groups, by Theorem 2.7, we may assume that f is an infra-nil-endomorphism.

Suppose $f : X \to X$ is an expanding infra-nil-endomorphism, then according to the definition, $X = G/\Gamma$ for some simply connected nilpotent Lie group G, $\Gamma \subset \mathrm{Aff}(G)$ is a discrete uniform subgroup, and f can be lifted to an automorphism A of G which respects Γ. A induces a linear map $a : \mathfrak{g} \to \mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ of G. f is expanding implies A is expanding and consequently a is an expanding linear map. Let $\Gamma' = G \cap \Gamma$, then Γ/Γ' is a finite group and $N = G/\Gamma'$ is a nilmanifold (see [AA] Theorem 1). Clearly N is a covering of X and f can be lifted to an expanding map $g : N \to N$. Thus we have the following commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{A} & G \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_1} \\ N & \xrightarrow{g} & N \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow{f} & X \end{array}$$

where both p_1 and p_2 are covering maps.

Now we will show that $g^* : H^l(N;\mathbb{R}) \to H^l(N;\mathbb{R})$ is expanding for any positive integer l. By Theorem 2.8, $H^*(N;\mathbb{R}) \cong H^*(\wedge\mathfrak{g}^*, \delta)$. Note that this isomorphism is natural and g descends from A or a. Thus the expanding property of g^* is equivalent to that

$$a^* : H^l(\wedge\mathfrak{g}^*) \to H^l(\wedge\mathfrak{g}^*)$$

is expanding for any $l > 0$.

By the construction of A, we know that a is a linear expanding map on the Lie algebra $\mathfrak{g}$. Then by Lemma 2.3, $a^* : \mathfrak{g}^* \to \mathfrak{g}^*$ is also expanding. Next it follows

from Lemma 2.4 that $a^* : \wedge^l \mathfrak{g} \to \wedge^l \mathfrak{g}$ is expanding for any $l > 0$. Thus according to Lemma 2.5, a induces expanding homomorphisms on the cohomology level.

Therefore we have proved that g^* is expanding on $H^l(N;\mathbb{R})$ for any positive integer l. Note that $H^*(N;\mathbb{R}) = \mathrm{Hom}(H_*(N;\mathbb{R});\mathbb{R})$, by Lemma 2.3, we know $g_* : H_l(N;\mathbb{R}) \to H_l(N;\mathbb{R})$ is also expanding for $l > 0$.

Taking homology on the above diagram, we get the commutative diagram

$$\begin{array}{ccc} H_l(N;\mathbb{R}) & \xrightarrow{g_*} & H_l(N;\mathbb{R}) \\ {\scriptstyle p_{2*}}\downarrow & & \downarrow{\scriptstyle p_{2*}} \\ H_l(X;\mathbb{R}) & \xrightarrow{f_*} & H_l(X;\mathbb{R}) \end{array}$$

By Lemma 2.9, p_{2*} is an isomorphism, then it follows from Lemma 2.2 that f_* is an expanding map for $l > 0$. At last, according to Lemma 2.3, $f^* : H^l(X;\mathbb{R}) \to H^l(X;\mathbb{R})$ is expanding as well. □

3 Proof of the main theorem

3.1 Maps between abelian groups

The first lemma is a useful fact about exact sequence in homological algebra. Except the elementary proof given here, it can also be proved in the language of category theory by using pull-back diagram. See Theorem II 6.2 and Lemma III 1.1 & 1.2 in [HS] for details.

Lemma 3.1. *Let A, B_1, B_2 and C be abelian groups and*

$$A \xrightarrow{\varphi=(i_1,i_2)} B_1 \oplus B_2 \xrightarrow{\psi=j_1-j_2} C$$

be an exact sequence, then

(a) i_1 is injective $\Rightarrow$ j_2 is injective.

(b) i_1 is injective $\Leftarrow$ j_2 is injective, provided that φ is injective.

(c) i_1 is surjective $\Rightarrow$ j_2 is surjective, provided that ψ is surjective.

(d) i_1 is surjective $\Leftarrow$ j_2 is surjective.

(e) All the above claims are still hold if we substitute i_1 and j_2 by i_2 and j_1 respectively.

Proof. Each statement can be shown by diagram chasing, for example, we prove (a) and (c) as follows.

(a) Suppose $j_2(b_2) = 0$, then $\psi(0, b_2) = 0$. So $(0, b_2) \in \mathrm{Im}\varphi$, we can find $a \in A$ such that $\varphi(a) = (0, b_2)$. It follows from the injectivity of i_1 that $a = 0$. Thus $b_2 = i_2(a) = 0$, which implies j_2 is injective.

(c) For any $c \in C$, as ψ is surjective, we may find $(b_1, b_2) \in B_1 \oplus B_2$ such that $\psi(b_1, b_2) = c$. By the surjectivity of i_1, pick up $a \in A$ such that $\varphi(a) = (-b_1, b_2')$. Now $j_2(b_2 + b_2') = \psi(0, b_2 + b_2') = \psi(b_1, b_2) + \psi(-b_1, b_2') = c + \psi \circ \varphi(a) = c$, which means j_2 is surjective.

(e) is clear by symmetry. $\square$

Lemma 3.2. *Let A, B be two finitely generated free abelian groups. Let $f : A \to A$ be an expanding homomorphism. Let $g : B \to B$ be an automorphism. Then there does not exist a non-zero homomorphism $h : A \to B$ such that $h \circ f = g \circ h$.*

Proof. Assume that there exists a non-zero homomorphism $h : A \to B$ such that $h \circ f = g \circ h$. The image of h, $h(A)$, is a subgroup of B. Thus it is a finitely generated free abelian group and there exists a basis $e_1, \dots, e_r$ of B and non-zero integers $a_1, \dots, a_s$ such that $a_1 e_1, \dots, a_s e_s$ form a basis of $h(A)$. Since f is expanding and $h \circ f = g \circ h$, the restriction of g to $h(A)$, $g|h(A) : h(A) \to h(A)$, is expanding (using the fact that A is isomorphic to the direct sum of $\ker h$ and $h(A)$). Let C denote the subgroup of B spanned by $e_1, \dots, e_s$. Since g maps $h(A)$ into $h(A)$, g maps C into C and the restriction of g to C, $g|C : C \to C$, is also expanding. The matrix M of g with respect to the basis $e_1, \dots, e_r$ looks like

$$\begin{pmatrix} M_1 & * \\ 0 & M_2 \end{pmatrix},$$

where M_1 is an $s \times s$ integer matrix and M_2 is an $(r-s) \times (r-s)$ integer matrix. Since $g|C : C \to C$ is expanding, the absolute value of the determinant of M_1 is greater than 1. Since M_2 is an integer matrix and non-degenerate (g is an automorphism, thus M is non-degenerate), the absolute value of the determinant of M_2 is greater than or equal to 1. Thus the absolute value of the determinant of M is greater than 1. But since g is an automorphism, the absolute value of the determinant of M equals 1, a contradiction. $\square$

3.2 Related results in algebraic topology

Most of the techniques in algebraic topology involved thereafter can be found in standard textbooks such as [Sp]. However for the readers convenience, we will recall some of them:

Theorem 3.3. *(Exact sequence for pair (X, A)) Let A be a subspace of a topological space X. there is a (co)homology long exact sequence associated to the pair (X, A):*

$$\cdots \to H_l(A) \xrightarrow{i} H_l(X) \xrightarrow{j} H_l(X, A) \xrightarrow{\partial} H_{l-1}(A) \xrightarrow{i} H_{l-1}(X) \cdots$$

$$\cdots \to H^l(X, A) \xrightarrow{j} H^l(X) \xrightarrow{i} H^l(A) \xrightarrow{\delta} H^{l+1}(X, A) \xrightarrow{j} H^{l+1}(X) \cdots$$

Theorem 3.4. *(Mayer-Vietoris sequence) Let X be a topological space and A, B be two subspaces whose interiors cover X. The Mayer-Vietoris sequence in singular homology for the triad (X, A, B) is a long exact sequence relating the singular homology groups of the spaces X, A, B, and the intersection $A \cap B$.*

$$\cdots \to H_{l+1}(X) \xrightarrow{\partial} H_l(A \cap B) \xrightarrow{(i_1,i_2)} H_l(A) \oplus H_l(B) \xrightarrow{j_1-j_2} H_l(X) \xrightarrow{\partial_*} H_{l-1}(A \cap B) \cdots$$

where the homomorphisms i_1, i_2, j_1, j_2 are induced from the inclusions $A \cap B \subset A$, $A \cap B \subset B$, $A \subset X$ and $B \subset X$ respectively.

Let $N \xrightarrow{\pi} X$ be a q-disk bundle and $\partial N \xrightarrow{\pi} X$ be the associated $(q-1)$-sphere bundle. Then a Thom class for the bundle is an element $t \in H^q(N, \partial N)$ such that the restriction of t to each fiber is non zero. If the Thom class exists, the disk bundle is called orientable. The Euler class e of the disk bundle or the associated sphere bundle is the image of t under the map $H^q(N, \partial N) \xrightarrow{j} H^q(N) \xrightarrow{(\pi^*)^{-1}} H^q(X)$.

Theorem 3.5. *(Naturality of Euler class) If $f : X \to X'$ is covered by an orientation preserving bundle map $\xi \to \xi'$, then $e(\xi) = f^*e(\xi')$.*

Theorem 3.6. *(Thom isomorphism theorem) Let $N \xrightarrow{\pi} X$ be an orientable q-disk bundle and $\partial N \xrightarrow{\pi} X$ be the associated $(q-1)$-sphere bundle with Thom class $t \in H^q(N, \partial N)$. Then the following homomorphisms are isomorphisms for all $l \in Z$*

$$\Phi^* : H^l(X) \to H^{l+q}(N, \partial N)$$

$$\Phi_* : H_{l+q}(N, \partial N) \to H_l(X)$$

where $\Phi^*(z) = \pi^*(z) \cup t$ and $\Phi_*(x) = \pi_*(t \cap x)$

From the Thom isomorphism theorem one can construct the Gysin sequence as follows:

Consider the commutative diagram

$$\begin{tikzcd}
H_{l+1}(N, \partial N) \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & H_l(\partial N) \arrow[r] \arrow[d, equal] & H_l(N) \arrow[r] \arrow[d, "\pi_*"] & H_l(N, \partial N) \arrow[r, "\partial"] \arrow[d, "\Phi_*"] & H_{l-1}(\partial N) \arrow[d, equal] \\
H_{l-q+1}(X) \arrow[r] & H_l(\partial N) \arrow[r] & H_l(X) \arrow[r, "\cap e"] & H_{l-q}(X) \arrow[r] & H_{l-1}(\partial N)
\end{tikzcd}.$$

The first line is the long exact sequence for the pair $(N, \partial N)$. The vertical map π_* is an isomorphism as $N \simeq X$. The vertical map Φ_* is also an isomorphism, called *Thom isomorphism.* Therefore the second line is also a long exact sequence, which is named as *Gysin sequence*. Moreover the map from $H_l(X)$ to $H_{l-q}(X)$ is defined by $x \mapsto e \cap x$, where e is the Euler class of the disk bundle ξ, $\cap$ means cap product.

3.3 Topology of sphere bundles

Lemma 3.7. *Let φ be an expanding map on the closed oriented manifold X and ξ be an oriented disk bundle $N = X \tilde{\times} D^q \xrightarrow{\pi} X$ such that φ can be lifted to a hyperbolic bundle embedding e on N. Then*

(a) the Euler class of ξ, $e(\xi) = 0 \in H^q(X)$;

(b) $H_l(\partial N) \cong H_l(X) \oplus H_{l-q+1}(X)$;

(c) Let $\{[c_1], [c_2], \cdots\}$ be a basis for $H_{l-q+1}(X)$, where each c_i is a cycle in X, then $[(\pi|\partial N)^{-1}(c_1)], [(\pi|\partial N)^{-1}(c_2)], \cdots$ are linearly independent in $H_l(\partial N)$. As a consequence, their span is a subspace $U \subset H_l(\partial N)$ with $\dim U = \dim H_{l-q+1}(X)$.

Proof. (a) We may assume that e is orientation preserving, otherwise we use e^2 instead of e. The embedding $e : X \tilde{\times} D^q \to X \tilde{\times} D^q$ is not a bundle map, however we can modify it to become a bundle map. Since ξ may not be a trivial bundle, we use local presentation of ξ. For each $x_0 \in X$, there is a neighborhood U of x such that $\xi|U$ is a trivial bundle, that is, $\xi|U \cong U \times D^q$. Similarly, there is a neighborhood V of $\varphi(x_0)$ such that $\xi|V$ is a trivial bundle, that is, $\xi|V \cong V \times D^q$. Possibly shrinking U, we may assume $\varphi(U) \subset V$. According to Definition 1.6, we may assume on $\xi|U \cong U \times D^q$,

$$e(x, y) = (\varphi(x), c(x) + \lambda r(x)y), x \in U, y \in D^q,$$

where $\lambda \in (0, 1)$, $c(x)$ and $r(x)$ are functions from U to D^q and $SO(q)$ respectively with $|c(x)| + \lambda < 1$. The the map defined by

$$(x, y) \to (\varphi(x), r(x)y)$$

gives a bundle map from $\xi|U$ to $\xi|V$ which is a lift of φ. This map is independent of the trivializations of ξ over U and V we chose. Thus we have a bundle map from N to N which is a lift of φ.

Thus by the naturality of Euler class, we have $\varphi^* e(\xi) = e(\xi)$. Note that by Theorem 1.4, φ^* is an expanding map on $H^q(X)$. So 1 cannot be an eigenvalue of φ^*, consequently $\varphi^* e(\xi) = e(\xi)$ implies $e(\xi) = 0$.

(b) In our current situation since the euler class $e = 0$, the Gysin sequence splits into short exact sequence

$$0 \to H_{l-q+1}(X) \xrightarrow{\rho} H_l(\partial N) \to H_l(X) \to 0. \tag{3.1}$$

Then $H_l(\partial N) \cong H_l(X) \oplus H_{l-q+1}(X)$ as all the homology groups here are over $\mathbb{R}$, and thus are vector spaces.

(c) We need to understand more about the map $\rho : H_{l-q+1}(X) \to H_l(\partial N)$ in (3.1). According to the geometric interpretation of cap product, for any $(l-q+1)$-cycle c in X, $\Phi_*^{-1}([c])$ is exactly represented by $\pi^{-1}(c)$, where $\Phi_* : H_{l+1}(N, \partial N) \to H_{l-q+1}(X)$ is the Thom isomorphism. Its boundary is $(\pi|\partial N)^{-1}(c)$, which represents $\rho([c]) \in H_l(\partial N)$. Now the result follows from the injectivity of ρ. $\square$

Lemma 3.8. *Let $f : X = B\tilde{\times}S^q \to Y$ be a map, where B is a finite CW-complex and Y is a $K(\pi, 1)$ space. If $q \geq 2$, then f is homotopic to $\bar{f} \circ \pi$, where $\pi : B\tilde{\times}S^q \to B$ is the projection, and $\bar{f} : B \to Y$.*

Proof. First we show by induction that f can be extended to a map $\tilde{f} : B\tilde{\times}D^{q+1} \to Y$ where $\partial(B\tilde{\times}D^{q+1}) = B\tilde{\times}S^q$.

Let B_i be the i-th dimensional skeleton of B, and $\tilde{X}_i = X \cup B\tilde{\times}D^{q+1}|_{B_i}$, where $i = 0, 1, \cdots, \dim B$. Since $q \geq 2$ and Y is $K(\pi, 1)$, clearly we can extend f to $\tilde{f}|\tilde{X}_0$. Suppose we have extended f to $\tilde{f}|\tilde{X}_{i-1}$. Then for each i-cell Δ_i in B, $\tilde{f}|$ has defined in $\partial(\Delta_i \times D^{q+1}) = (\partial\Delta_i \times D^{q+1} \cup \Delta_i \times S^q) \cong S^{i+q}$, and still by the $K(\pi, 1)$ property of Y, we may extend $\tilde{f}|\partial(\Delta_i \times D^{q+1})$ to $\Delta_i \times D^{q+1}$. (Here we just write $\Delta_i \times D^{q+1}$ because any disk bundle on Δ_i is trivial.) Therefore $\tilde{f}$ can be extended to $\tilde{X}_i$ after finitely many such steps.

Then the lemma follows easily. $\square$

3.4 $\Omega(f) =$ ***DE attractors*** implies $M = \mathbb{Q}$-***homology sphere***

Now we are ready to prove

Theorem 1.1. *If there exists a diffeomorphism $f : M \to M$ on a closed, oriented n-manifold M such that $\Omega(f)$ consists of finitely many oriented $(\pm)$ attractors derived from expanding maps, then M is a rational homology sphere. Moreover all those attractors are of type $(n-2, 2)$.*

If $\Omega(f)$ consists of finitely many DE attractors, we know $\Omega(f)$ must be the union of two disjoint DE attractors S_1 and S_2, one is the attractor of f and the other is

the attractor of f^{-1} (see [JNW] Lemma 1). Suppose the two defining disk bundles of attractors are $N_1 \cong X_1^{p_1} \tilde{\times} D^{q_1}$ and $N_2 \cong X_2^{p_2} \tilde{\times} D^{q_2}$ respectively. We have

$$S_1 = \bigcap_{h=1}^{\infty} f^h(N_1),\ S_2 = \bigcap_{h=1}^{\infty} f^{-h}(N_2),$$

Lemma 3.9. *Suppose an orientable manifold N is a disk bundle on a closed oriented manifold X. Let the embedding $e: N \to \mathrm{Int}(N)$ be the lift of an expanding map on X, $N' = e(N)$, $K = N \backslash \mathrm{Int}(N')$. For any integer l, the two maps $H_l(\partial N) \to H_l(K) \leftarrow H_l(\partial N')$, induced by the inclusions $\partial N \subset K, \partial N' \subset K$ respectively are isomorphisms. (cf. Figure 1)*

Figure 1: The sketch of self embedding of a disk bundle

Proof. First, consider the Mayer-Vietoris long exact sequence for the pair (N', K). We know

$$H_l(N') \oplus H_l(K) \xrightarrow{\psi} H_l(N) \xrightarrow{\partial} H_{l-1}(\partial N')$$

is exact. $N \simeq X$ implies $H_*(N) \cong H_*(X)$ through the projection map. Then it follows from Lemma 2.9 that the map from $H_l(N')$ to $H_l(N)$ is surjective and thus the first map ψ is surjective. By the exactness, the second map ∂ must be 0 for any l and we get the short exact sequence

$$0 \to H_l(\partial N') \xrightarrow{\varphi=(i_1,i_2)} H_l(N') \oplus H_l(K) \xrightarrow{\psi=j_1-j_2} H_l(N) \to 0.$$

By Lemma 2.9, j_1 is an isomorphism, then according to Lemma 3.1, the map $i_2 : H_l(\partial N') \to H_l(K)$ is also an isomorphism. Half is done.

Next, consider the long exact sequence of the pair $(K, \partial N')$:

$$\cdots \to H_l(\partial N') \xrightarrow{i_2} H_l(K) \to H_l(K, \partial N') \to H_{l-1}(\partial N') \xrightarrow{i_2} H_{l-1}(K) \cdots$$

Note that i_2 involved are isomorphisms, so $H_l(K, \partial N') = 0$. Then by Poicaré duality and algebraic duality, we have

$$0 = H_l(K, \partial N') = H^{n-l}(K, \partial N) = \mathrm{Hom}(H_{n-l}(K, \partial N), \mathbb{R}).$$

Hence $H_{n-l}(K, \partial N) = 0$ for all l, or equivalently $H_l(K, \partial N) = 0$ for any integer l.

Finally, consider the long exact sequence for the pair $(K, \partial N)$:

$$\cdots \to H_{l+1}(K, \partial N) \to H_l(\partial N) \to H_l(K) \to H_l(K, \partial N) \to \cdots .$$

It then follows from $H_{l+1}(K, \partial N) = H_l(K, \partial N) = 0$ that $H_l(\partial N) \to H_l(K)$ is an isomorphism. $\square$

The setting in the above lemma appears in a single attractor, while the next lemma reveals the homological relations between the two DE attractors in the non-wandering set. Recall that our assumption is $S_1 = \bigcap_{h=1}^{\infty} f^h(N_1)$, $S_2 = \bigcap_{h=1}^{\infty} f^{-h}(N_2)$. Then $\bigcup_{h=1}^{\infty} f^{-h}(\mathrm{Int} N_1) = M - S_2$, it follows that $f^k(\partial N_2) \subset N_1$, $M \backslash N_1 \subset f^k(N_2)$, for some large integer k. Without loss of generality, we can substitute N_2 by $f^k(N_2)$. Then

$$M = N_1 \cup N_2, \ P := N_1 \cap N_2 \text{ with } \partial P = \partial N_1 \cup \partial N_2.$$

Say $n = \dim M$, then the base manifold X_1 of N_1 is at most $n-2$ dimensional (cf. Remark 1.8). Hence $H_{n-1}(N_1) = 0$, which implies that ∂N_2 separates N_1 into two components. One of them is P which contains ∂N_1, while the other part P' should contain $f^k(N_1)$ for some large integer k. Set $N' = f^k(N_1)$, $K = N_1 \backslash \mathrm{Int}(N')$. All the notations introduced above are illustrated in Figure 2.

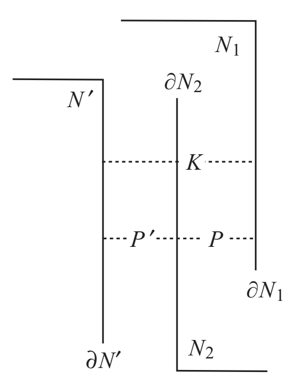

Figure 2: The sketch of two intersecting disk bundles

Lemma 3.10. *For any integer l, the two maps $H_l(\partial N_1) \to H_l(P) \leftarrow H_l(\partial N_2)$, induced by the inclusions $\partial N_1 \subset P, \partial N_2 \subset P$ respectively are isomorphisms.*

In particular, $\beta_l(\partial N_1) = \beta_l(\partial N_2)$ for any integer l.

Proof. Applying the Mayer-Vietoris sequence for the pair (P', P), we get the exact sequence

$$H_l(\partial N_2) \xrightarrow{(i_1,i_2)} H_l(P') \oplus H_l(P) \xrightarrow{j_1-j_2} H_l(K).$$

The inclusion $\partial N_1 \subset P \subset K$ induces $H_l(\partial N_1) \to H_l(P) \stackrel{j_2}{\to} H_l(K)$. By lemma 3.9, this composition is an isomorphism. Hence j_2 is surjective. Similarly, it follows from the inclusion $\partial N' \subset P' \subset K$ and Lemma 3.9 that j_1 is surjective. Moreover, by Lemma 3.1(d),(e), i_2 is also surjective. Then the surjectivity of i_2 and j_2 gives the inequality

$$\dim H_l(\partial N_2) \geq \dim H_l(P) \geq \dim H_l(K) = \dim H_l(\partial N_1). \tag{3.2}$$

If we consider the diffeomorphism f^{-1} instead, then by symmetry, the above inequality turns into

$$\dim H_l(\partial N_1) \geq \dim H_l(\partial N_2). \tag{3.3}$$

Combining (3.2) and (3.3), we get

$$\dim H_l(\partial N_1) = \dim H_l(P) = \dim H_l(K) = \dim H_l(\partial N_2). \tag{3.4}$$

Now all the homology groups in $H_l(\partial N_1) \to H_l(P) \stackrel{j_2}{\to} H_l(K)$ have the same dimension and the composition is an isomorphism by Lemma 3.9, thus the map $H_l(\partial N_1) \to H_l(P)$ has to be an isomorphism. By symmetry $H_l(\partial N_2) \to H_l(P)$ is also an isomorphism. □

Consider the Mayer-Vietoris sequence for the pair (N_1, N_2) :

$$H_l(P) \xrightarrow{\varphi_l=(r_1,r_2)} H_l(N_1) \oplus H_l(N_2) \xrightarrow{\psi_l=s_1-s_2} H_l(M). \tag{3.5}$$

Lemma 3.11. *φ_l is surjective for all $l > 0$.*

Proof. Otherwise ψ_l is not zero for some $l > 0$, and then at least one s_i, say s_1, is not a zero map. As $f(N_1) \subset N_1$, we have the following commutative diagram

$$\begin{tikzcd} H_l(N_1;\mathbb{Z}) \arrow[r, "s_1"] \arrow[d, "(f|N_1)_*"'] & H_l(M;\mathbb{Z}) \arrow[d, "f_*"] \\ H_l(N_1;\mathbb{Z}) \arrow[r, "s_1"'] & H_l(M;\mathbb{Z}) \end{tikzcd}.$$

Note that $f|N_1$ is a lift of an expanding map on the base manifold X_1 whose induced map on $H_l(X_1;\mathbb{Z})$ is expanding by Theorem 1.4 for $l>0$. Hence $(f|N_1)_*: H_l(N_1;\mathbb{Z}) \to H_l(N_1;\mathbb{Z})$ is expanding as well. On the other hand, it follows from $f: M \to M$ is a diffeomorphism that $f_*: H_l(M;\mathbb{Z}) \to H_l(M;\mathbb{Z})$ is an isomorphism. But s_1 is non-zero, which contradicts to Lemma 3.2. $\square$

Proof of Theorem 1.1: Recall that the two defining disk bundles are

$$N_1 \cong X_1^{p_1} \tilde{\times} D^{q_1}, N_2 \cong X_2^{p_2} \tilde{\times} D^{q_2},$$

then $n = p_1 + q_1 = p_2 + q_2$. By symmetry, assume that $2 \leq q_1 \leq q_2$. then $n-2 \geq p_1 \geq p_2 \geq 1$.

Case 1: $q_2 \geq 3$.

Since X_j is a closed orientable manifold of dimension p_j, we have $\beta_{p_j}(X_j) = 1$, $(j=1,2)$. Set $l = p_1$, then $1 = \beta_l(X_1) \geq \beta_l(X_2)$.

Consider the diagram

$$\begin{tikzcd} H_l(P) \arrow[r, "\varphi_l"] & H_l(N_1) \oplus H_l(N_2) \ , \\ U_2 \subset H_l(\partial N_2) \arrow[u, "\cong"', "i_2"] \arrow[ur, "\phi_2"'] & \end{tikzcd}$$

where φ_l is defined in (3.5), i_2 is induced by inclusion $\partial N_2 \subset P$, U_2 is given by Lemma 3.7 (c), and $\phi_2 = \varphi_l \circ i_2$.

Since X_j admits an expanding map, the universal cover of X_j is contractible by Theorem 2.6 (a), hence $N_j \simeq X_j$ is a $K(\pi,1)$ space. Note that ϕ_2 is induced by inclusion $g_j : \partial N_2 \to N_j$. According to Lemma 3.7 (c), U_2 has a basis $\{[c_1 \tilde{\times} S^{q_2-1}], [c_2 \tilde{\times} S^{q_2-1}], \cdots\}$, where $c_1, c_2, \cdots$ are $l - q_2 + 1$ cycles in X_2. By Lemma 3.8, as $q_2 - 1 \geq 2$, g_j can be homotoped so that the image of $c_1 \tilde{\times} S^{q_2-1}$, $c_2 \tilde{\times} S^{q_2-1}$, $\cdots$ under g_j are all of dimension $l - q_2 + 1$,$(j=1,2)$. Thus ϕ_2 maps all these basis elements to zero in both $H_l(N_1)$ and $H_l(N_2)$, in other words, $U_2 \subset \ker\phi_2$.

It follows from Lemma 3.7 that $H_l(\partial N_2) \cong H_l(X_2) \oplus H_{l-q_2+1}(X_2)$ while $\dim U_2 = \dim H_{l-q_2+1}(X_2)$. Thus

$$\dim \mathrm{Im}\phi_2 \leq \beta_l(X_2) < \beta_l(X_1) + \beta_l(X_2) = \beta_l(N_1) + \beta_l(N_2).$$

Consequently ϕ_2 cannot be surjective.

On the other hand, i_2 is an isomorphism by Lemma 3.10, φ_l is surjective by Lemma 3.11, so ϕ_2 must be surjective. We derive a contradiction.

Case 2: $q_1 = q_2 = 2$.

Consider

$$H_l(X_j) \oplus H_{l-1}(X_j) \xrightarrow{\cong} H_l(P) \xrightarrow{\varphi_l} H_l(N_1) \oplus H_l(N_2) \xrightarrow{\cong} H_l(X_1) \oplus H_l(X_2), j = 1, 2,$$

where the left isomorphism comes from Lemma 3.7 (b) and Lemma 3.10, and the right isomorphism comes from $N_j \simeq X_j$.

The surjectivity of φ_l from Lemma 3.11 implies that

$$\beta_l(X_j) + \beta_{l-1}(X_j) \geq \beta_l(X_1) + \beta_l(X_2), j = 1, 2, \tag{3.6}$$

or equivalently $\beta_{l-1}(X_1) \geq \beta_l(X_2)$ and $\beta_{l-1}(X_2) \geq \beta_l(X_1)$. Let l goes from 1 to $n-2$, we get

$$1 = \beta_0(X_j) \geq \cdots \geq \beta_{n-3}(X_2) \geq \beta_{n-2}(X_1) = 1$$

and

$$1 = \beta_0(X_k) \geq \cdots \geq \beta_{n-3}(X_1) \geq \beta_{n-2}(X_2) = 1$$

Therefore $\beta_i(X_1) = \beta_i(X_2) = 1$ for $i = 0, ..., n-2$.

Now the equality holds in (3.6) for $l = 1, \cdots, n-2$, hence in the whole Mayer-Vietoris sequence of (N_1, N_2) (3.5), φ_l is an isomorphism, $\psi_l = 0$ for $l = 1, \cdots, n-2$, therefore

$$H_{n-1}(M; \mathbb{R}) = \cdots = H_1(M; \mathbb{R}) = 0.$$

It follows from the universal theorem for homology that

$$H_{n-1}(M; \mathbb{Q}) = \cdots = H_1(M; \mathbb{Q}) = 0,$$

i.e. M is a rational homology sphere. □

Proof of Corollary 1.2. Suppose $f : M \to M$ is a diffeomorphism such that $\Omega(f)$ consists of finitely many oriented DE attractors. Let N_1, N_2 be defined as in the beginning of the proof of Theorem 1.1.

(b) By Theorem 1.1, $N_1 \cong T^{n-2} \tilde{\times} D^2$ and $N_2 \cong T^{n-2} \tilde{\times} D^2$. Thus $\beta_1(N_1) = \beta_1(N_2) = n-2$, and by Lemma 3.7 (b), $\beta_1(\partial N_1) = \beta_1(\partial N_2) = n-1$. Now we have

$$\mathbb{R}^{n-1} = H_1(P) \xrightarrow{\varphi_1 = (r_1, r_2)} H_1(N_1) \oplus H_1(N_2) = \mathbb{R}^{2n-4}.$$

Then φ_1 can not be surjective when $n \geq 4$, which contradicts to Lemma 3.11. We proved (b).

(a) When $n = 4$, by Theorem 1.1, the dimension of the base manifolds X_j must be 2. By Theorem 2.7, X_j is covered by Nil-manifold. Since the only orientable closed 2-manifold covered by Nil-manifold is torus, $j = 1, 2$. So (a) follows from (b).

(c) is included in (a) and Theorem 1.1. □

Acknowledgement. The first three authors are partially supported by grant No.10631060 of the National Natural Science Foundation of China, the second author is also partially supported by NSFC project 60603004. The last author would like to thank Professor R. Kirby for his continuous support and encouragement.

References

[AA] A. L. Auslander, *Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups*, Annals of Math. **71** (1960), 579-590.

[CE] C.Chevalley, S. Eilenberg, *Cohomology theory of Lie groups and Lie algebras*, Trans. Amer. Math. Soc. **63** (1948), 85-124.

[ES] D. Epstein, M. Shub, *Expanding endomorphisms of flat manifolds*, Topology **7**(1968), 139–141.

[FW] J. Franks; B. Williams, *Anomalous Anosov flows.* Global theory of dynamical systems. 158–174, Lecture Notes in Math. **819**, Springer, Berlin, 1980.

[Gr] M. Gromov, *Groups of polynomial growth and expanding maps.* Inst. Hautes tudes Sci. Publ. Math. No. **53**(1981), 53-73.

[HS] P.J. Hilton, U. Stammbach, *A course in homological algebra*, Grad. Texts in Math. Vol **4**, Springer-Verlag, 1971.

[JNW] B. J. Jiang, Y. Ni, S. C. Wang, *3-manifolds that admit knotted solenoids as attractors*, Trans. Amer. Math. Soc. **356** (2004), 4371-4382.

[JWZ] B. Jiang, S. Wang, H. Zheng, *No embeddings of solenoids into surfaces.* Proc. Amer. Math. Soc. **136**(2008), no. 10, 3697–3700.

[Ma] J. Mather, *Characterization of Anosov diffeomorphisms*, Nederl. Akad. van Wetensch. Proc. Ser. A, Armsterdam 71 = Indag. Math. **30**(1968), No.5 .

[No] K. Nomizu, *On the cohomology of compact homogeneous spaces of nilpotent Lie groups. Ann.* of Math. (2) 59, (1954). 531–538.

[Sh1] M. Shub, *Endomorphisms of compact differentiable manifolds*, Amer. J. Math. **91**(1969), 175-199.

[Sh2] M. Shub, *Expanding maps* 1970 Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968) 273–276

[Sm] S. Smale, *Differentiable dynamical systems*, Bull. Amer. Math. Soc. **73** (1967), 747–817.

[Sp] E.H. Spanier, *Algebraic topology*, Springer-Verlag, 1966.

[Vi] L. Vietoris, *Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen,* (German) Math. Ann. **97** (1927), no. 1, 454–472.

[WZ] S.C. Wang, Q. Zhou, *Any 3-manifold 1-dominates at most finitely many geometric 3-manifolds,* Math. Ann. **322**(2002), no. 3, 525-535.

School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China *E-mail:* dingfan@math.pku.edu.cn

Institute of Mathematics, Academy of Mathematics and Systems science, Chinese Academy of Sciences, Beijing 100080 P. R. China *E-mail:* pjz@amss.ac.cn

School of Mathematical Sciences, Peking University, Beijing 100871, P.R.China *E-mail:* wangsc@math.pku.edu.cn

Department of Mathematics, University of California at Berkeley, CA 94720, USA *E-mail:* jgyao@math.berkeley.edu